JOURNAL OF APPROXIMATION THEORY 75, 157–166 (1993)

Convergence Rates of Ergodic Limits and Approximate Solutions

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Communicated by Paul L. Butzer

Received November 28, 1991; accepted in revised form June 2, 1992

This paper is concerned with the convergence rates of two processes $\{A_\alpha\}$ and $\{B_\alpha\}$, under the assumption that $\|A_\alpha\| = O(1)$ and there is a closed operator A such that $B_\alpha A \subset AB_\alpha = I - A_\alpha$, $\|AA_\alpha\| = O(e(\alpha))$, and $B_\alpha^* x^* = \varphi(\alpha) x^*$ for $x^* \in R(A)^\perp$, where $e(\alpha) \to 0$ and $|\varphi(\alpha)| \to \infty$. It was previously proved that $\{A_\alpha\}$ converges strongly on $N(A) \oplus \overline{R(A)}$ to P, the projection onto $N(A)$ along $\overline{R(A)}$, and $\{B_\alpha\}$ converges strongly on $A(D(A) \cap \overline{R(A)})$ to A_1^{-1}, the inverse operator of $A_1 = A \mid \overline{R(A)}$. In this paper, the two processes are shown to be saturated with order $O(e(\alpha))$, and their saturation classes are characterized. The result provides a unified approach to convergence rates for many particular mean ergodic theorems and for various methods of solving the equation $Ax = y$. We discuss in particular applications to integrated semigroups, cosine operator functions, and tensor product semigroups.

1. INTRODUCTION

The mean ergodic theorem with rates, due to Butzer and Westphal [3, 4], for Cesàro means of powers of a power-bounded linear operator T on a Banach space X states that for $x \in X_0 := N(I-T) \oplus \overline{R(I-T)}$ one has $\|n^{-1} \sum_{k=0}^{n-1} T^k x - Px\| = O(1/n)$ [resp. $o(1/n)$] $(n \to \infty)$ if and only if $x \in N(I-T) \oplus [(I-T)\overline{R(I-T)}]^{\sim}_{X_o}$ [resp. $x \in N(I-T)$]. Here P denotes the projection on $N(I-T)$ parallel to $\overline{R(I-T)}$, and $[Y]^{\sim}_{X_o}$ means the completion of a Banach subspace Y of X_o relative to X_o. Analogous results for Abel means of $\{T^n\}$ as well as for means of a (C_0)-semigroup of operators were also obtained in [3, 2].

Recently, we formulated [12–14] a fairly general form of abstract mean ergodic theorem (see Theorem A), which not only subsumes many particular mean ergodic theorems, but also provides an appropriate way of finding approximate solutions of functional equations of the form $Ax = y$.

[1] Research supported in part by the National Science Council of the Republic of China.

0021-9045/93 $5.00

Working within this framework we now intend to equip ergodic limits and approximate solutions with an order of approximation. The general result is stated in Section 2 and proved in Section 3. Applications to particular examples, such as integrated semigroups, cosine operator functions, and tensor product semigroups, are then included in Section 4.

2. Main Results

To begin with we state the abstract mean ergodic theorem as follows.

Let X be a Banach space and $B(X)$ be the Banach algebra of all bounded linear operators on X. Let $A: D(A) \subset X \to X$ be a closed linear operator, and let $\{A_\alpha\}$ and $\{B_\alpha\}$ be two nets in $B(X)$ satisfying:

(C1) $\|A_\alpha\| \leqslant M$ for all α;

(C2) $R(B_\alpha) \subset D(A)$ and $B_\alpha A \subset AB_\alpha = I - A_\alpha$ for all α;

(C3) $R(A_\alpha) \subset D(A)$ for all α, and $\|AA_\alpha\| = O(e(\alpha))$ with $\lim_\alpha e(\alpha) = 0$.

(C4) $B_\alpha^* x^* = \varphi(\alpha)\, x^*$ for all $x^* \in R(A)^\perp$, and $|\varphi(\alpha)| \to \infty$.

Note that (C2) implies $A_\alpha A \subset AA_\alpha$ for all α. The function $e(\alpha)$ in (C3) is to act as an estimator of the convergence rates of $\{A_\alpha x\}$ and $\{B_\alpha y\}$, approximating respectively the ergodic limit and the solution of $Ax = y$, in practical applications. The assumption (C4) plays a key role in the proof of our theorems and prevails among practical examples.

The abstract mean ergodic theorem for the systems $\{A_\alpha\}$ and $\{B_\alpha\}$ is proved in [12, Thm. 1.1, Coro. 1.4, and Remark 1.7]. For reference we state it in the following.

Theorem A. *Let A, $\{A_\alpha\}$, and $\{B_\alpha\}$ be as assumed above.*

(i) *The set $D(P)$ of all $x \in X$ for which the limit $Px := \lim_\alpha A_\alpha x$ exists is precisely equal to $N(A) \oplus \overline{R(A)}$, which is a closed linear subspace of X, and the operator P thus defined is a bounded linear projection with the range $R(P) = N(A)$ and the null space $N(P) = \overline{R(A)}$.*

(ii) *The set $D(B)$ of all $y \in X$ for which the limit $By := \lim_\alpha B_\alpha y$ exists is precisely equal to $A(D(A) \cap \overline{R(A)})$, and By is the unique solution of the functional equation $Ax = y$ in $\overline{R(A)}$; in other words, the operator B thus defined is the inverse operator A_1^{-1} of the restriction $A_1 := A \mid \overline{R(A)}$ of A to $\overline{R(A)}$, which is a closed operator with range $R(B) = D(A_1) = D(A) \cap \overline{R(A)}$, and domain $D(B) = R(A_1) = A(D(A) \cap \overline{R(A)})$.*

(iii) *$D(P) = X$ if and only if $D(B) = R(A)$. These two identities hold in particular when X is reflexive.*

Let $X_0 := D(P) = N(A) \oplus \overline{R(A)}$ and $X_1 := \overline{R(A)}$. Since the operator $B: D(B) \subset X_1 \to X_1$ is closed, its domain $D(B)$ $(= R(A_1))$ is a Banach space with respect to the norm $\|x\|_B := \|x\| + \|Bx\|$. The completion of $D(B)$ relative to X_1, denoted by $[D(B)]^{\sim}_{X_1}$ $(= [R(A_1)]^{\sim}_{X_1})$, is the set of all those $y \in X_1$ for which there exist a sequence $\{y_n\} \subset D(B)$ and a constant $K > 0$ such that $\|y_n\|_B \leqslant K$ for all n and $\|y_n - y\| \to 0$. Using it, we try to characterize the Favard (or saturation) classes for the two processes $\{A_\alpha\}$ and $\{B_\alpha\}$.

THEOREM 1. *Let the hypotheses of Theorem A be satisfied. Then for* $y \in X$ *we have*:

(1) *If* $\|B_\alpha y\| = o(1)$, *then* $y = 0$.

(2) $\|B_\alpha y\| = O(1)$ *if and only if* $y \in [R(A_1)]^{\sim}_{X_1}$.

THEOREM 2. *Let A be a closed operator and $\{A_\alpha\}$ and $\{B_\alpha\}$ be two nets in $B(X)$ which satisfy conditions* (C1)–(C4) *and the condition*:

(C5) $\|A_\alpha y\| = O(e(\alpha))$ [*resp.* $o(e(\alpha))$] *implies* $\|B_\alpha y\| = O(1)$ [*resp.* $o(1)$].

Then the following are true:

(1) *For* $x \in X_0$ *one has* $\|A_\alpha x - Px\| = o(e(\alpha))$ *if and only if* $x \in N(A)$.

(2) *For* $x \in X_0$ *one has* $\|A_\alpha x - Px\| = O(e(\alpha))$ *if and only if* $x \in N(A) \oplus [R(A_1)]^{\sim}_{X_1}$.

(3) *If* $y \in R(A_1)$ *and* $\|B_\alpha y - A_1^{-1} y\| = o(e(\alpha))$, *then* $y = 0$.

(4) *For* $y \in R(A_1)$ *one has* $\|B_\alpha y - A_1^{-1} y\| = O(e(\alpha))$ *if and only if* $A_1^{-1} y \in [R(A_1)]^{\sim}_{X_1}$, *or equivalently*, $y \in A(D(A) \cap [R(A_1)]^{\sim}_{X_1})$.

When $\{A_\alpha\}$ is mean ergodic, i.e., $D(P) = X$, by (iii) of Theorem A we have $R(A_1) = D(B) = R(A)$. When X is reflexive one even has $[R(A_1)]^{\sim}_{X_1} = [D(B)]^{\sim}_{X_1} = D(B) = R(A)$. This follows from the weak closedness of B and weak sequential precompactness of bounded sets in a reflexive space. (cf. also Coro. 1.8 of [12]).

In [13, Thm. 2], it has been proved that if $X = L_1(\mu)$ with μ a σ-finite measure, and if $\{A_\alpha\}$ and $\{B_\alpha\}$ satisfy (C1) with $M = 1$, (C2), (C3), and (C4), then $\|B_\alpha y\| = O(1)$ is equivalent to $y \in R(A)$. From this and (2) of Theorem 1, we infer that $[R(A_1)]^{\sim}_{X_1}$ is identical to $R(A)$ in this case too. Thus we can formulate the following corollary.

COROLLARY 3. *If, in addition to the hypotheses of Theorem* 2, *it is assumed that X is a reflexive Banach space* [*resp.* $X = L_1(\mu)$ *with μ a σ-finite measure and A_α's are contractions*], *then the following assertions hold*:

(1) *For* $x \in X$ [*resp.* $x \in X_0$] *one has* $\|A_\alpha x - Px\| = O(e(\alpha))$ *if and only if* $x \in N(A) \oplus R(A)$.

(2) $\|B_\alpha y\| = O(1)$ *if and only if* $y \in R(A)$.

(3) *For* $y \in R(A)$ *one has* $\|B_\alpha y - A_1^{-1} y\| = O(e(\alpha))$ *if and only if* $y \in R(A^2)$.

Remark. It is worthwhile mentioning that if A has closed range, then one has $\|A_\alpha - P\| = O(e(\alpha))$, $\|B_\alpha \mid R(A)\| = O(1)$, and $\|B_\alpha \mid R(A) - A_1^{-1}\| = O(e(\alpha))$, no matter whether X is reflexive or not. Actually, these four conditions are equivalent to each other (see [10]).

3. Proofs

Proof of Theorem 1. (1) is an obvious consequence of (ii) of Theorem A.

To prove (2), we first suppose $y \in X$ be such that $\|B_\alpha y\| = O(1)$. Put $J_\alpha := I - A_\alpha$. Since for every $x^* \in R(A)^\perp$ we have, by (C4),

$$\|B_\alpha y\| \, \|x^*\| \geq |\langle B_\alpha y, x^* \rangle| = |\langle y, B_\alpha^* x^* \rangle| = |\varphi(x)| \, |\langle y, x^* \rangle|,$$

the boundedness of $\{B_\alpha y\}$ and the assumption: $|\varphi(\alpha)| \to \infty$ imply that $\langle y, x^* \rangle = 0$. Hence $y \in {}^\perp(R(A)^\perp) = \overline{R(A)}$ $(= X_1)$ and so $\|J_\alpha y - y\| = \|A_\alpha y\| \to \|Py\| = 0$, by (i) of Theorem A. Moreover, by (C2) and Theorem A(ii) one sees that $B_\alpha y \in D(A) \cap \overline{R(A)}$, $J_\alpha y = AB_\alpha y \in A(D(A) \cap \overline{R(A)}) = D(B)$, and $BJ_\alpha y = BAB_\alpha y = B_\alpha y$. Hence we have

$$\begin{aligned} \|J_\alpha y\|_B &= \|J_\alpha y\| + \|BJ_\alpha y\| = \|y - A_\alpha y\| + \|B_\alpha y\| \\ &\leq (1 + \|A_\alpha\|) \, \|y\| + \|B_\alpha y\| = O(1). \end{aligned}$$

This shows that y belongs to $[D(B)]^{\sim}_{X_1} = [R(A_1)]^{\sim}_{X_1}$.

Conversely, if $y \in [D(B)]^{\sim}_{X_1}$, there exists a sequence $\{y_n\}$ in $D(B)$ such that $\|y_n\|_B \leq K$ for all n and $\|y_n - y\| \to 0$. It follows by (ii) of Theorem A, (C2), and (C1) that

$$\begin{aligned} \|B_\alpha y_n\| &= \|B_\alpha ABy_n\| = \|(I - A_\alpha) \, By_n\| \leq (1 + \|A_\alpha\|) \, \|By_n\| \\ &\leq (1 + \|A_\alpha\|) \, \|y_n\|_B \leq (1 + \|A_\alpha\|) \, K = O(1). \end{aligned}$$

Passing to the limit we obtain that $\|B_\alpha y\| = O(1)$.

Proof of Theorem 2. Since every x in X_0 has a unique representation $x = y + z$ with $y \in \overline{R(A)}$ and $z \in N(A)$, by (C2) and (i) of Theorem A one has $A_\alpha z = z = Px$ so that $A_\alpha x - Px = A_\alpha y$. If $\|A_\alpha x - Px\| = o(e(\alpha))$ [resp.

$O(e(\alpha))$], then by (C5) one has $\|B_\alpha y\| = o(1)$ [resp. $O(1)$]. Then (1) [resp. (2)] of Theorem 1 implies that $y = 0$ [resp. $y \in [R(A_1)]^{\sim}_{X_1}$]. This proves the "only if" parts of assertions (1) and (2). The "if" part of (1) follows from (i) of Theorem A and (C2).

To show the "if" part of (2), let $y \in [R(A_1)]^{\sim}_{X_1} = [D(B)]^{\sim}_{X_1}$ and let $\{y_n\}$ be a sequence in $D(B)$ such that $\|y_n\|_B \leqslant K$ for all n and $\|y_n - y\| \to 0$. Then we have

$$\|A_\alpha y_n\| = \|A_\alpha A B y_n\| = \|A A_\alpha B y_n\| \leqslant \|A A_\alpha\| \ \|B y_n\|$$
$$\leqslant \|A A_\alpha\| \ \|y_n\|_B \leqslant \|A A_\alpha\| \ K,$$

and hence $\|A_\alpha y\| \leqslant \|A A_\alpha\| \ K = O(e(\alpha))$. Therefore, if $x \in N(A) \oplus [R(A_1)]^{\sim}_{X_1}$, then $\|A_\alpha x - Px\| = \|A_\alpha y\| = O(e(\alpha))$.

Finally, since application of (C2) and (ii) of Theorem A yields

$$B_\alpha y - By = B_\alpha A B y - By = (B_\alpha A - I)\, By = -A_\alpha B y$$

for $y \in D(B)$, if $y \in R(A_1)$ ($\subset \overline{R(A)}$) and $\|B_\alpha y - A_1^{-1} y\| = o(e(\alpha))$, then if follows from (1) that $A_1^{-1} y \in N(A)$. Thus $A_1^{-1} y \in N(A) \cap \overline{R(A)} = \{0\}$ and hence $y = 0$. This proves (3). Similarly, assertion (4) follows from (2).

4. Examples

4.1. *Abelian Ergodic Theorem with Rates*

Let A be a closed operator such that $0 \in \overline{\rho(A)}$ and such that $\|\lambda(\lambda - A)^{-1}\| = O(1)(\lambda \to 0,\ \lambda \in \rho(A))$. Set $A_\lambda := \lambda(\lambda - A)^{-1}$ and $B_\lambda := -(\lambda - A)^{-1}$, $\lambda \in \rho(A)$. Clearly $\{A_\lambda\}$ and $\{B_\lambda\}$ satisfy conditions (C1)–(C5), with $e(\lambda) = |\lambda| \to 0$ and $\varphi(\lambda) = \lambda^{-1} \to \infty$ as $\lambda \to 0$ (cf. [12, Ex. V]). Then it follows from Theorem A that $N(A) \cap \overline{R(A)} = \{0\}$, and $X_0 = N(A) \oplus \overline{R(A)}$ is a closed subspace of X. Let P be the projection onto $N(A)$ parallel to $\overline{R(A)}$, and let $A_1 := A \mid \overline{R(A)}$. The following theorem follows from Theorem 1 and 2 immediately.

THEOREM 4. *Let A be a closed operator such that $0 \in \overline{\rho(A)}$ and $\|\lambda(\lambda - A)^{-1}\| = O(1)(\lambda \to 0)$. Then the following are true*:

(1) *For $x \in X_0$, one has $\|\lambda(\lambda - A)^{-1} x - Px\| = o(|\lambda|)(\lambda \to 0)$ if and only if $x \in N(A)$.*

(2) *For $x \in X_0$, one has $\|\lambda(\lambda - A)^{-1} x - Px\| = O(|\lambda|)(\lambda \to 0)$ if and only if $x \in N(A) \oplus [R(A_1)]^{\sim}_{X_1}$.*

(3) *If $\|(\lambda - A)^{-1} y\| = o(1)(\lambda \to 0)$, then $y = 0$.*

(4) $\|(\lambda - A)^{-1} y\| = O(1)(\lambda \to 0)$ *if and only if* $y \in [R(A_1)]^{\sim}_{X_1}$.

(5) *If* $y \in A(D(A) \cap \overline{R(A)})$ *and* $\|(\lambda - A)^{-1} y + A_1^{-1} y\| = o(|\lambda|)(\lambda \to 0)$, *then* $y = 0$.

(6) *For* $y \in A(D(A) \cap \overline{R(A)})$, *one has* $\|(\lambda - A)^{-1} y + A_1^{-1} y\| = O(|\lambda|)(\lambda \to 0)$ *if and only if* $y \in A[D(A) \cap [R(A_1)]^{\sim}_{X_1}]$.

4.2. *n-Times Integrated Semigroups*

A C_0-semigroup of operators in $B(X)$ is called a 0-times integrated semigroup, and for $n > 0$ a strongly continuous family $\{T(t); t \geqslant 0\}$ in $B(X)$ is called an n-times integrated semigroup (cf. [1, 5, 7–9, 16]) if $T(0) = 0$ and

$$T(t)\, T(s) = \frac{1}{(n-1)!} \left(\int_0^{t+s} - \int_0^t - \int_0^s \right) (t + s - r)^{n-1}\, T(r)\, dr \qquad (s, t \geqslant 0).$$

If $T(\cdot)$ is exponentially bounded, i.e., there are $M \geqslant 0$ and $w \in R$ such that $\|T(t)\| \leqslant Me^{wt}$ for all $t \geqslant 0$, and if it is nondegenerate in the sense that $x = 0$ whenever $T(t)\, x = 0$ for all $t > 0$, then there exists a unique closed operator A satisfying $(w, \infty) \subset \rho(A)$ and $(\lambda - A)^{-1} x = \int_0^\infty \lambda^n e^{-\lambda t} T(t)\, x\, dt$ for all $x \in X$ and $\lambda > w$. This operator A is called the generator of $T(\cdot)$; it is not necessarily densely defined.

Let $A_t := (n+1)!\, t^{-n-1} \int_0^t T(s)\, ds$ and $B_t := -(n+1)!\, t^{-n-1} \times \int_0^t \int_0^s T(u)\, du\, ds$ for $t > 0$. If $\|T(t)\| = O(t^n)(t \to \infty)$, then $\{A_t\}$ and $\{B_t\}$ satisfy conditions (C1)–(C4), with $e(t) = t^{-1}$ and $\varphi(t) = -t/(n+2)$ (cf. [13, 14]). One can also easily verify that $\|A_t y\| = O(t^{-1})$ [resp. $o(t^{-1})$] implies $\|B_t y\| = O(1)$ [resp. $o(1)$]. That is, (C5) is satisfied. Hence Theorems A, 1, and 2 can be applied to $\{A_t\}$ and $\{B_t\}$. On the other hand, the assumption $\|T(t)\| \leqslant Mt^n$, $t \geqslant 0$, implies that the generator A satisfies $(0, \infty) \subset \rho(A)$ and $\|\lambda(\lambda - A)^{-1}\| = O(1)(\lambda \to 0^+)$ so that Theorem 4 works.

Strong and uniform ergodic theorems for $T(\cdot)$ have been given in [14]. The following theorem is about the convergence rates of ergodic limits and of approximate solutions of $Ax = y$; it follows from Theorems 1, 2, and 4.

THEOREM 5. *Let* $\{T(t); t \geqslant 0\}$ *be a nondegenerate n-times integrated semigroup with generator* A, *and suppose* $\|T(t)\| \leqslant Mt^n$ *for all* $t \geqslant 0$.

(1) *If* $x \in N(A) \oplus \overline{R(A)}$, *the following assertions are equivalent*:

(a) $\|(n+1)!\, t^{-n-1} \int_0^t T(s)\, x\, ds - Px\| = O(1/t)(t \to \infty)$,

(b) $\|\lambda(\lambda - A)^{-1} x - Px\| = O(\lambda)(\lambda \to 0^+)$,

(c) $x \in N(A) \oplus [R(A_1)]^{\sim}_{X_1}$.

(2) *For any* $y \in X$, *the following assertions are equivalent*:

(d) $\|t^{-n-1} \int_0^t \int_0^s T(u)\, y\, du\, ds\| = O(1)(t \to \infty)$,

(e) $\|(\lambda - A)^{-1} y\| = O(1)(\lambda \to 0^+)$,

(f) $y \in [R(A_1)]^{\sim}_{X_1}$.

(3) *If* $y \in A(D(A) \cap \overline{R(A)})$, *the following assertions are equivalent*:

(g) $\|(n+1)!\, t^{-n-1} \int_0^t \int_0^s T(u)\, y\, du\, ds + A_1^{-1} y\| = O(1/t)(t \to \infty)$,

(h) $\|(\lambda - A)^{-1} y + A_1^{-1} y\| = O(\lambda)(\lambda \to 0^+)$,

(i) $y \in A(D(A) \cap [R(A_1)]^{\sim}_{X_1})$.

Remarks. (I) There are generators of n-times integrated semigroups of order $O(t^n)$, that do not generate semigroups of class (C_0). For example, the Laplacian on $L^p(R)$ $(1 \leqslant p \leqslant \infty)$ [7].

(II) When $n = 0$, parts (1) and (2) of Theorem 5 reduce to Butzer and Dickmeis' result [2] on (C_0)-semigroups. Similar application of Theorems 1 and 2 to discrete semigroups (cf. [12, Ex. II]) will reproduce the cited result of Butzer and Westphal [3].

4.3. *Cosine Operator Functions*

A strongly continuous family $\{C(t); t \geqslant 0\}$ in $B(X)$ is called a cosine operator function if $C(0) = I$ and $C(t+s) + C(t-s) = 2C(t)\, C(s)$, $t \geqslant s \geqslant 0$ (cf. [6, 15]). The generator A, defined by $Ax := \lim_{t \to 0^+} 2t^{-2}(C(t) - I)\, x$, is a densely defined closed operator. Suppose that $\|C(t)\| \leqslant M$ for all $t \geqslant 0$. Then $(0, \infty) \subset \rho(A)$ and

$$\lambda(\lambda^2 - A)^{-1} x = \int_0^\infty e^{-\lambda t} C(t)\, x\, dt \qquad (x \in X, \lambda > 0).$$

Thus $\|\lambda(\lambda - A)^{-1}\| \leqslant M$ for $\lambda > 0$ so that Theorem 4 can be applied.

For $t > 0$ let

$$A_t := 2t^{-2} \int_0^t \int_0^s C(u)\, du\, ds$$

and

$$B_t := -2t^{-2} \int_0^t \int_0^s \int_0^u \int_0^v C(w)\, dw\, dv\, du\, ds.$$

Then we have $B_t A \subset A B_t = I - A_t$ and $A A_t = 2t^{-2}(C(t) - I)$, so that conditions (C1)–(C4) are satisfied with $e(t) = t^{-2}$ and $\varphi(t) = t^2/12$. It is also easy to check that $\|B_t y\| = O(1)$ whenever $\|A_t y\| = O(t^{-2})(t \to \infty)$, i.e., (C5) holds.

Strong convergence and uniform convergence of $\{A_t x\}$ and $\{B_t y\}$ as $t \to \infty$ have been discussed in [12, p. 440] and [14, Thm. 6], respectively. The convergence rates are estimated in the following theorem, which is a specialization of Theorems 1, 2, and 4.

THEOREM 6. *Let $\{C(t); t \geq 0\}$ be a uniformly bounded cosine operator function with generator A. We have*:

(1) *If $x \in N(A) \oplus \overline{R(A)}$, the following assertions are equivalent*:

(a) $\|A_t x - Px\| = O(1/t^2)(t \to \infty)$,

(b) $\|\lambda(\lambda - A)^{-1} x - Px\| = O(\lambda)(\lambda \to 0^+)$,

(c) $x \in N(A) \oplus [R(A_1)]^{\sim}_{X_1}$.

(2) *For given $y \in X$, the following assertions are equivalent*:

(d) $\|B_t y\| = O(1)(t \to \infty)$,

(e) $\|(\lambda - A)^{-1} y\| = O(1)(\lambda \to 0^+)$,

(f) $y \in [R(A_1)]^{\sim}_{X_1}$.

(3) *If $y \in A(D(A) \cap \overline{R(A)})$, the following assertions are equivalent*:

(g) $\|B_t y - A_1^{-1} y\| = O(1/t^2)(t \to \infty)$,

(h) $\|(\lambda - A)^{-1} y + A_1^{-1} y\| = O(\lambda)(\lambda \to 0^+)$,

(i) $y \in A(D(A) \cap [R(A_1)]^{\sim}_{X_1})$.

4.4. *Tensor Product Semigroups*

For $i = 1, 2$, let X_i be a Banach space and $\{T_i(t); t \geq 0\} \subset B(X_i)$ be a (C_0)-semigroup with the infinitesimal generator A_i. Suppose $\|T_i(t)\| \leq M_i e^{w_i t}$, $t \geq 0$, $i = 1, 2$. The family $\{S(t); t \geq 0\}$ of operators on $B(X_2, X_1)$, defined by $S(t) E = T_1(t) E T_2(t)(E \in B(X_2, X_1))$, is a semigroup in $B(B(X_2, X_1))$, and is called the tensor product semigroup of $T_1(\cdot)$ and $T_2(\cdot)$. The generator Δ of $S(\cdot)$, defined by the strong operator limit $\Delta E := so - \lim_{t \to 0^+} t^{-1}(S(t) E - E)$, is closed and densely defined relative to the strong (and also the weak) operator topology; it is precisely the operator which has as its domain the set of all those $E \in B(X_2, X_1)$ for which $ED(A_2) \subset D(A_1)$ and $A_1 E + EA_2$ is bounded on $D(A_2)$, and sends each such E to $\overline{A_1 E + EA_2}$. For $\lambda > w_1 + w_2$, $\lambda - \Delta$ is invertible and

$$(\lambda - \Delta)^{-1} Ex = \int_0^\infty e^{-\lambda t}(S(t) E) x \, dt \qquad (E \in B(X_2, X_1), x \in X_2)$$

(cf. [10–12, 9]). If $w_1 + w_2 \leq 0$, then $(0, \infty) \subset \rho(\Delta)$ and $\|\lambda(\lambda - \Delta)^{-1}\| \leq M_1 M_2$ for all $\lambda > 0$, so that Theorem 4 can be applied to Δ.

For $t > 0$ define the operators A_t and B_t by

$$(A_t E) x := t^{-1} \int_0^t (S(s) E) x \, ds = t^{-1} \int_0^t T_1(s) E T_2(s) x \, ds$$

and

$$(B_t E)x := -t^{-1}\int_0^t\int_0^s (S(u)E)x\,du\,ds = -t^{-1}\int_0^t\int_0^s T_1(u)ET_2(u)x\,du\,ds$$

for $E\in B(X_2, X_1)$ and $x\in X_2$. It is known [12] that $B_t\Delta\subset\Delta B_t = t^{-1}(T(t)-I)$, and $A_t\Delta\subset\Delta A_t = t^{-1}(S(t)-I)$, and $B_t^* x^* = (-1/2)\,tx^*$ for $x^*\in R(\Delta)^\perp$. Thus if $w_1+w_2\leqslant 0$, then $\{A_t\}$, $\{B_t\}$ satisfy conditions (C1)–(C5), with $e(t)=t^{-1}$ and $\varphi(t)=(-1/2)\,t$.

A mean ergodic theorem for $S(\cdot)$ is proved in [10], and the approximate solution of the operator equation $A_1E+EA_2=F$ is studied in [11, 12]. We now apply Theorems 1, 2, and 4 to give rates of convergence for ergodic limits and approximate solutions.

THEOREM 7. *Suppose that $w_1+w_2\leqslant 1$, and let $\Pi: N(\Delta)\oplus\overline{R(\Delta)}\to N(\Delta)$ be the projection with $R(\Pi)=N(\Delta)$ and $N(\Pi)=\overline{R(\Delta)}$, where the overbar denotes the uniform operator closure. We have:*

(1) *If $E\in N(\Delta)\oplus\overline{R(\Delta)}$, the following assertions are equivalent:*

(a) $\|t^{-1}\int_0^t T_1(s)ET_2(s)\,ds-\Pi E\| = O(1/t)(t\to\infty)$,

(b) $\|\lambda(\lambda-\Delta)^{-1}E-\Pi E\| = O(\lambda)(\lambda\to 0^+)$,

(c) $E\in N(\Delta)\oplus[\Delta(D(\Delta)\cap\overline{R(\Delta)})]^{\sim}_{\overline{R(\Delta)}}$.

(2) *For given $F\in B(X_2, X_1)$ the following assertions are equivalent:*

(d) $\|t^{-1}\int_0^t\int_0^s T_1(u)FT_2(u)\,du\,ds\| = O(1)(t\to\infty)$,

(e) $\|(\lambda-\Delta)^{-1}F\| = O(1)(\lambda\to 0^+)$,

(f) $F\in[\Delta(D(\Delta)\cap\overline{R(\Delta)})]^{\sim}_{\overline{R(\Delta)}}$.

(3) *If $F\in\Delta(D(\Delta)\cap\overline{R(\Delta)})$, the following assertions are equivalent:*

(g) $\|t^{-1}\int_0^t\int_0^s T_1(u)FT_2(u)\,du\,ds+(\Delta\mid\overline{R(\Delta)})^{-1}F\| = O(1/t)(t\to\infty)$,

(h) $\|(\lambda-\Delta)^{-1}F+(\Delta\mid\overline{R(\Delta)})^{-1}F\| = O(\lambda)(\lambda\to 0^+)$,

(i) $F\in\Delta\{D(\Delta)\cap[\Delta(D(\Delta)\cap\overline{R(\Delta)})]^{\sim}_{\overline{R(\Delta)}}\}$.

Remarks. (I) It is known that if X_1 is reflexive, then (d) is equivalent to $F\in R(\Delta)$ (see [11, Coro. 3.6]). Hence in this case the set $[\Delta(D(\Delta)\cap\overline{R(\Delta)})]^{\sim}_{\overline{R(\Delta)}}$ is identical to $R(\Delta)$, so that (c) can be replaced by (c'): $E\in N(\Delta)\oplus R(\Delta)$, and (i) can be replaced by (i'): $F\in R(\Delta^2)$.

(II) We know [10, 11] that a tensor product semigroup is just a (Y)-semigroup on $B(X_2, X_1)$ for some suitable subspace Y of $(B(X_2, X_1))^*$. The same argument as above will lead to a similar theorem for Y-semigroups, whose formulation is like the $n=0$ case of Theorem 5. Furthermore, because the integral of a Y-semigroup becomes a once integrated semigroup (see [14, p. 410] and also [9, p. 153] for Δ being the generator of a once integrated semigroup), one can apply Theorem 5, case $n=1$, to obtain rates for $(C, 2)$-means of a (Y)-semigroup, and particularly, of a tensor product semigroup.

Acknowledgment

The author thanks the referees for some helpful suggestions which make this paper more informative.

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